ON LINEAR DIFFERENTIAL EQUATIONS WITH EXPONENTIAL COEFFICIENTS AND STATIONARY LAGS IN THE ARGUMENT. IRREGULAR CASE

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Considered are linear differential equations with periodic and almost periodic coefficients and with stationary lags [delays] in the argument. The application of the Laplace transform leads to the solution of linear Diophantine equations. The terms of the series of the transforms constitute semigroups for which there is established an isomorphism with the group of certain generalized numbers.

This isomorphism simplifies the computations and makes it possible to investigate the stability of the quasi-stationary equations. In particular, an asymptotic criterion for stability of the solutions of a second order linear differential equation, with almost periodic coefficients is obtained.

1. We shall consider the following system of linear differential equations [1]

$$\sum_{q=0}^{l} e^{-\alpha qt}\Big(A_{qn} \frac{d^nY(t)}{dt^n} + \sum_{k=0}^{n-1} \int_{-h}^{0} dA_{qk}(\vartheta) \frac{d^kY(t+\vartheta)}{dt^k} \Big) = \Phi(t) \qquad (1.1)$$

Here Y is an m-dimensional vector; the A_{qn} are constant $m \times m$ matrices satisfying the conditions

$$A_{on} = E, \qquad \sum_{q=1}^{l} |A_{qn}| < 1 \qquad (1.2)$$

where E is the unit matrix, and $|A|$ denotes the norm (1.2) of the matrix A. The number l is assumed to be finite.

The elements $a_{sj}^{qk}(\vartheta)$ of the matrix $A_{qk}(\vartheta) = \| a_{sj}^{qk}(\vartheta) \|_1^m$ are assumed to be functions of bounded variation [2] on $[-h, 0]$, $(h > 0)$. The integrals in (1.1) are Stieltjes integrals [2].

We shall assume* that

$$\alpha_0 \equiv 0, \qquad \operatorname{Re}\alpha_q \equiv 0 \qquad (q = 1, \ldots, l) \tag{1.3}$$

Among the numbers $\operatorname{Im}\alpha_q$ there may be non-commensurate numbers. Suppose that the transform of a vector $\Phi(t)$ $(t \geqslant 0)$ is the vector $Q(p)$, which is regular and bounded when $\operatorname{Re} p \geqslant b = \text{const}$. We are looking for a solution $Y(t)$, with $t > 0$, of the system (1.1) that will satisfy, when $t \in [-h, 0]$, the initial conditions

$$Y(t) = Y_0^{(0)}(t), \ldots, d^{n-1}Y(t) / dt^{n-1} = Y_0^{(n-1)}(t) \tag{1.4}$$

Here it is sufficient to assume that the vectors $Y_0^{(j)}$ $(j = 0, 1, \ldots, n-1)$ are absolutely integrable on $[-h, 0]$. The vectors $Y(t), \ldots, d^{n-1}Y(t)/dt^{n-1}$ are assumed to be continuous from the right at the point $t = 0$.

Denoting the Laplace [3] transform of $Y(t)$ by $F(p)$, we have

$$F(p) = \int_0^\infty Y(t) e^{-pt} dt \tag{1.5}$$

Multiplying the terms of the system (1.1) by e^{-pt} and integrating the results with respect to t from 0 to $+\infty$, we obtain the following system of linear difference equations for the vector $F(p)$

$$F(p) = \sum_{q=1}^{l} K_q(p) F(p + \alpha_q) + \Omega(p) \tag{1.6}$$

Here we have introduced the following notation [1]

$$K_q(p) = -L_0^{-1}(p) L_q(p + \alpha_q), \qquad \Omega(p) = L_0^{-1}(p) R(p) \tag{1.7}$$

The matrices $L_q(p)$ and the vector $R(p)$ are known [1]

$$L_q(p) = A_{qn}p^n + \sum_{k=0}^{n-1} p^k \int_{-h}^{0} e^{p\vartheta} dA_{qk}(\vartheta) \qquad (q = 0, 1, \ldots, l) \tag{1.8}$$

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* The general case $\operatorname{Re}\alpha_q \geqslant 0$ in (1.1) can be reduced to the case (1.3).

$$R(p)=Q(p)+\sum_{q=0}^{l}A_{qn}\sum_{j=0}^{n-1}Y_0^{(j)}(0)(p+\alpha_q)^{n-j-1}+$$
$$+\sum_{q=0}^{l}\sum_{j=0}^{n-2}\sum_{k=j+1}^{n-1}\int_{-h}^{0}e^{(p+\alpha_q)\vartheta}dA_{qk}(\vartheta)\,Y_0^{(j)}(0)(p+\alpha_q)^{k-j-1}-$$
$$-\sum_{q=0}^{l}\sum_{k=0}^{n-1}\int_{-h}^{0}\int_{t}^{0}e^{(p+\alpha_q)(\vartheta-t)}dA_{qk}(\vartheta)\,Y_0^{(k)}(t)\,dt \tag{1.9}$$

We shall point out the most important properties of $L_q(p)$, and $\Omega(p)$; i.e.

$$p^{-n}L_q(p)\to A_{qn},\quad \Omega(p)\to 0\quad \text{when } \operatorname{Re} p\to+\infty \tag{1.10}$$

The convergence is uniform in Im p. From (1.2) it follows that if Re $p \geqslant b$, where b is a sufficiently large number, then the solution $F(p)$ of the equation (1.6) can be obtained by the method of successive approximations [2, p.45]. This yields [1]

$$F(p)=\Omega(p)+\sum_{\sigma=1}^{\infty}\sum_{q_j=1,\ldots,l}K_{q_1}(p)\,K_{q_2}(p+\alpha q_1)\times K_{q_3}(p+\alpha q_1+\alpha q_2)\times$$
$$\times\ldots\times K_{q\sigma}(p+\alpha q_1+\alpha q_2+\ldots+\alpha q_{\sigma-1})\,\Omega(p+\alpha q_1+\alpha q_2+\ldots+\alpha q_\sigma) \tag{1.11}$$

The series (1.11) for $F(p)$ converges when Re $p\geqslant b$. The original $Y(t)$ obtained from the series (1.11) will be a series that converges absolutely and uniformly when $0\leqslant t\leqslant T<\infty$. For a system of differential equations with almost periodic coefficients, this series will differ but little from the series obtained in [3]. The series (1.1) and its original do not yield directly a way for solving the problem on the stability of the solution of (1.1).

2. In this section we indicate a relationship between the investigation of the system (1.1) and the study of linear Diophantine equations.

Let us associate in a one-to-one way the generalized number $[\chi,\sigma]$ (χ, σ are non-negative integers) with the product of the matrices (1.7) of the form

$$K_{q_1}(p)\,K_{q_2}(p+\alpha_{q_1})\,K_{q_3}(p+\alpha_{q_1}+\alpha_{q_2})\ldots K_{q_\sigma}(p+\alpha_{q_1}+\alpha_{q_2}+\ldots+\alpha_{q_\sigma}) \tag{2.1}$$

$$\chi=(q_1-1)+(q_2-1)\,l+(q_3-1)\,l^2+\ldots+(q_\sigma-1)\,l^{\sigma-1}$$
$$\chi\leqslant l^\sigma-1 \tag{2.2}$$

We let the unit matrix E correspond to the number [0,0]. We shall denote this correspondence by a double-headed arrow $\leftrightarrow$; thus we have, e.g. $K_2(p)K_1(p+\alpha_2) \leftrightarrow [1,2]$.

The indicated correspondence (isomorphism) is one-to-one, since every integer χ can be expressed in a unique way in the scale of notation to the base l. The sums and differences of products of matrices of the type (2.1) will correspond to numbers $[\chi, \sigma]$ connected by plus and minus signs.

Example 2.1. Let us find the matrix which will correspond to the sum

$$[4,2]+[1,2]+[14,2]+[11,2]+\ldots \quad (l=4) \tag{2.3}$$

Expressing the numbers 4, 1, 14, 11 in the scale of notation to the base [radix] 4, we obtain from (2.1) and (2.2)

$$K_1(p)\,K_2(p+\alpha_1)+K_2(p)\,K_1(p+\alpha_2)+K_3(p)\,K_4(p+\alpha_3)+K_4(p)\,K_3(p+\alpha_4)+\ldots \tag{2.4}$$

In order to preserve the correspondence (2.1), (2.2), it is necessary to give a rule for multiplying the numbers $[\chi, \sigma]$. This rule is the following non-commutative relation

$$[\chi_1, \sigma_1]\,[\chi_2, \sigma_2] = [\chi_1+\chi_2 l^{\sigma_1}, \sigma_1+\sigma_2] \tag{2.5}$$

From (2.5) it follows that any number $[\chi, \sigma]\,(\sigma > 0)$ that corresponds to (2.1), can be represented as the product

$$[\chi, \sigma] = [q_1-1, 1]\,[q_2-1,1]\ldots[q_\sigma-1,1] = \left[\sum_{k=1} (q_k-1)\,l^{k-1}, \sigma\right] \tag{2.6}$$

With every number $[\chi, \sigma]$ we associate the numbers $[\chi, \sigma]^{(\gamma)}$ of the form

$$[\chi, \sigma]^{(\gamma)} = \left[\sum_{k=1}^{\sigma-\gamma} (q_k-1)\,l^{k-1}, \sigma-\gamma\right] \qquad (\gamma = 1, \ldots, \sigma-1) \tag{2.7}$$

These numbers will be called the derived numbers for $[\chi, \sigma]$. By definition we write

$$[\chi, \sigma]^{(0)} \equiv [\chi, \sigma], \qquad [\chi, \sigma]^{(\sigma)} \equiv [0, 0] \tag{2.8}$$

Let us consider the function $\alpha([\chi, \sigma])$, defined in the following way for the number $[\chi, \sigma]$ corresponding to (2.1)

$$\alpha([\chi, \sigma]) = \alpha_{q_1}+\alpha_{q_2}+\ldots+\alpha_{q_\sigma}, \qquad \alpha([0, 0]) \equiv 0 \tag{2.9}$$

From the definition of multiplication (2.5) and the expression (2.9) for the function α, follows the fundamental property of $\alpha([\chi, \sigma])$:

$$\alpha([\chi_1, \sigma_1]\ [\chi_2, \sigma_2]) = \alpha([\chi_2, \sigma_1]) + \alpha([\chi_2, \sigma_2]) \tag{2.10}$$

From (2.7) and (2.10) it follows that each of the numbers $\alpha([\chi, \sigma]^{(\gamma)})$ ($\gamma = 0, 1, \ldots, \sigma$) can differ from the neighboring one only by the quantity $\alpha_q (q = 1, \ldots, l)$.

Let us order the set of numbers

$$\alpha_{k_1, k_2, \ldots, k_l} = k_1\alpha_1 + k_2\alpha_2 + \ldots + k_l\alpha_l \qquad (k_1, k_2, \ldots, k_l = 0, 1, 2, \ldots) \tag{2.11}$$

We shall say that the number $\alpha_{k_1, k_2, \ldots, k_l}$ precedes the number $\alpha_{k_1', k_2', \ldots, k_l'}$ if $k_1 + k_2 + \ldots + k_l < k_1' + k_2' + \ldots + k_l'$, and when $k_1 + k_2 + \ldots + k_l = k_1' + k_2' + \ldots + k_l'$, if the first non-zero difference $k_1 - k_1'$, $k_2 - k_2'$, $\ldots$, $k_l - k_l'$ is positive. Among the numbers (2.11), there can occur numbers which are equal in numerical value. Therefore we have to renumber them without repetition and without omitting numerical values. Let us denote them by $\beta_r (r = 0, 1, 2, \ldots)$, $\beta_0 = 0$.

Let us consider the sum s_r of all the distinct numbers satisfying the equation

$$\alpha([\chi, \sigma]) = \beta_r \qquad (r = 0, 1, 2, \ldots) \tag{2.12}$$

Equation (2.9) shows that (2.12) is a linear Diophantine equation, while $[\chi, \sigma]$ depends, according to (2.2), on the order of the numbers α_q.

The sum of matrices, corresponding to s_r of the form (2.1), we shall denote by $S_r(p)$, $S_r(p) \leftrightarrow s_r$. Making use of these notations, we can rewrite the series (1.11) in the form

$$F(p) = \sum_{r=0}^{\infty} S_r(p)\, \Omega(p + \beta_r) \equiv \sum_{r=0}^{\infty} S_r(p) L_0^{-1}(p + \beta_r) R(p + \beta_r) \tag{2.13}$$

3. Let us study the matrix $S_0(p)L_0^{-1}(p)$ separately. In the following section it will be shown that the singularities of this matrix can determine the asymptotic behavior of the solution $Y(t)$ of the system (1.1). For the most important case, when the coefficients of the equation (1.1) are real, one can find for every number $\alpha_q \neq 0$, a number $\alpha_q' = -\alpha_q$, ($\mathrm{Re}\,\alpha_q = 0$). This implies that into the sum $S_r(p)$ there will enter as terms products of matrices of the type (2.1), which have poles of arbitrarily high order at the points $p = p_{k_0, k_1, \ldots, k_l}$. Here we use the notation

$$p_{k_0, k_1, \ldots, k_l} = \rho_{k_0} - k_1 a_1 - k_2 a_2 - \ldots - k_l a_l$$
$$(k_q = 0, 1, 2, \ldots, \; q = 0, 1, \ldots, l) \tag{3.1}$$

The numbers ρ_0, ρ_1, ρ_2, ... are the roots of the equation

$$\operatorname{Det} L_0 (p) = 0 \tag{3.2}$$

This follows from (1.11) and the definition (1.7) for $K_q(p)$. The denumerable set ρ_0, ρ_1, ρ_2, ... can be arranged in the order of decreasing absolute values

$$\operatorname{Re} \rho_0 \geqslant \operatorname{Re} \rho_1 \geqslant \operatorname{Re} \rho_2 \geqslant \ldots, \qquad \operatorname{Re} \rho_n \underset{n \to \infty}{\to} -\infty \tag{3.3}$$

Let us now consider the equation (2.12) that defines s_0

$$\alpha([\chi, \sigma]) = 0 \tag{3.4}$$

If the number $[\chi_1, \sigma_1] \in s_0$ (that is, if the number $[\chi_1, \sigma_1]$ enters into the sum s_0) and if $[\chi_2, \sigma_2] \in s_0$, then it follows from (2.10) that $[\chi_1, \sigma_1]$ $[\chi_2, \sigma_2] \in s_0$. This means that the solutions $[\chi, \sigma]$ of equation (3.4) form a multiplicative semigroup [5], which we shall denote by W. The semigroup W is a subsemigroup of the entire multiplicative group [5] of the numbers $[\chi, \sigma]$ with the law of multiplication given by (2.5).

A number $[\chi, \sigma] \in s_0$ will be called a simple solution of the equation (3.4) if $\alpha([\chi, \sigma]^{(\gamma)}) \neq 0$ $(\gamma = 1, \ldots, \sigma - 1)$. The sum of all the simple solutions $[\chi, \sigma]$ of equation (3.4) we shall denote by s_0^*.

A number $[\chi, \sigma] \in s_0$ will be called a compound solution of the equation (3.4) if zero occurs among the numbers $\alpha([\chi, \sigma]^{(\gamma)})(\gamma = 1, 2, \ldots, \sigma - 1)$. Every compound solution of the equation (3.4) can be expressed in a unique manner as the product of two, three, or more simple solutions of equation (3.4). The number [0,0] will be defined to be a compound number. The greater the number of simple solutions in the expansion of a number $[\chi, \sigma]$, the greater will be the number of meromorphic factors $L_0^{-1}(p)$ in (2.1), and the higher will be the order of the poles of the expression (2.1) at the points $p = \rho_j$ defined by (3.2). The matrices (2.1) that correspond to the simple solutions s_0^*, contain only one factor $L_0^{-1}(p)$.

One can say that the numbers entering into s_0^*, and the number [0,0] constitute a generating set [5, p.139] of the semigroup W. All solutions $[\chi, \sigma]$ of equation (3.4) that enter into s_0 can be obtained in the following way from the generating set s_0^*:

$$s_0 = [0, 0] + s_0^* + s_0^* s_0^* + s_0^* s_0^* s_0^* + \ldots = [0, 0] + s_0^* s_0 \tag{3.5}$$

For the corresponding matrix expressions we obtain

$$S_0(p) = F + S_0^*(p) S_0(p), \qquad S(\rho) = (E - S_0^*(p)]^{-1} \tag{3.6}$$

Let us introduce the matrix $D(p)$ with the aid of (1.8)

$$D(p) = L_0(p) - L_0(p) S_0^*(p) \tag{3.7}$$

We see that $S_0(p)L_0^{-1}(p) = D^{-1}(p)$. From the above considerations we obtain the following expansion of the type (1.11) for $S_0^*(p)$:

$$S_0^*(p) = \sum_{\sigma=2}^{\infty} \sum_{\varkappa_\sigma} K_{q_1}(p) K_{q_1}(p + \alpha_{q_1}) \ldots K_{q_\sigma}(p + \alpha_{q_1} + \ldots + \alpha_{q_{\sigma-1}}) \tag{3.8}$$

Here $\varkappa_\sigma$ stands for the expressions $q_1, q_2, \ldots, q_l = 1, 2, \ldots, l$, where the q_j satisfy the conditions

$$\alpha_{q_1} + \alpha_{q_2} + \ldots + \alpha_{q_\sigma} = 0 \tag{3.9}$$

$$0 \overline{\in} \{\alpha_{q_1}, \alpha_{q_1} + \alpha_{q_2}, \alpha_{q_1} + \alpha_{q_2} + \alpha_{q_3}, \ldots, \alpha_{q_1} + \alpha_{q_2} + \ldots + \alpha_{q_{\sigma-1}}\} \tag{3.10}$$

For the case $\alpha_1 = -\alpha_2 (l = 2)$ in (1.1), the theory of generalized numbers $[\chi, \sigma]$ has been applied in [6] for a complete analytic continuation of $F(p)$ in (1.11) to the entire complex plane p.

4. We shall apply the results of section 3 to the investigation of the stability of a system of equations simpler than that of (1.1) with a small parameter μ

$$\frac{d^n Y(t)}{dt^n} + \sum_{k=0}^{n-1} \int_{-h}^{0} dA_{0k}(\vartheta, \mu) \frac{d^k Y(t+\vartheta)}{dt^k} +$$

$$+ \mu \sum_{q=1}^{l} e^{-\alpha_q t} \left(A_{qn}(\mu) \frac{d^n Y(t)}{dt^n} + \sum_{k=0}^{n-1} \int_{-h}^{0} dA_{qk}(\vartheta, \mu) \frac{d^k Y(t+\vartheta)}{dt^k} \right) = 0 \tag{4.1}$$

The elements of the matrix $A_{qk}(\vartheta, \mu)$ are assumed to be differentiable a sufficient number of times with respect to μ if $0 \leqslant \mu \leqslant \mu_1$. When $\mu = 0$, the system of equation (4.1) degenerates into a system with constant coefficients and with a stationary lag in the argument. In this case all terms, except the meromorphic matrix $S_0(p)L_0^{-1}(p)R(p)|_{\mu=0}$, will disappear in the series (2.13) and (1.11). The matrix $S_0(p) \equiv E$ when $\mu = 0$. The vector $R(p)$ (1.9) has for its elements the entire functions $Q(p) \equiv 0$. When $\mu > 0$ is sufficiently small, only the matrix $S_0(p)L_0^{-1}(p)$ can have poles with coefficients in the principal parts of the expansion. From (3.6), (3.7) it follows that the poles which determine the asymptotic behavior of the solutions of the system (4.1), can be found by means of the equation

$$\operatorname{Det} D(p) \equiv \operatorname{Det}(L_0(p) - L_0(p) S_0^*(p)) = 0 \tag{4.2}$$

The matrices which occur in (4.2) are defined in (1.8) and (3.8).

The formula (4.2) can be solved directly in one important particular case.

Let $\rho_0(\mu)$, $\rho_1(\mu)$, $\rho_2(\mu)$, ... be the roots of the equation (3.2) which has been constructed for the system (4.1). They are assumed to be continuous in $0 \leqslant \mu \leqslant \mu_1$.

Suppose that for one of these roots $\rho^*(\mu)$ the following condition is satisfied

$$\rho_k(0) - \rho^*(0) \neq \beta_r \qquad (r = 1, 2, \ldots, \quad k = 0, 1, 2, \ldots) \tag{4.3}$$

From (3.8) it follows that the singular points of the terms of the series $L_0(p)S_0^*(p)$ can occur only at the points

$$p_{k,r} = \rho_k(\mu) - \beta_r \qquad (k = 0, 1, 2, \ldots, r = 1, 2, 3, \ldots) \tag{4.4}$$

where $\beta_r \neq 0$ when $r \neq 0$. If the numbers α_1, α_2, ..., α_l are commensurate, i.e. if $\alpha_q = n_q\theta i$, $\operatorname{Im}\theta = 0$, n_q an integer, then $|\beta_r| \geqslant \theta > 0$ ($r = 1$, 2, ...). Hence, for sufficiently small $\mu_2 \geqslant \mu > 0$, $\varepsilon > 0$, the matrix $L_0(p)S_0^*(p)$ will be analytic in the circle $|p - \rho^*(0)| \leqslant \varepsilon$, and will have a norm which is arbitrarily small. The series (3.8) will converge absolutely and uniformly when $|p - \rho^*(0)| \leqslant \varepsilon$, $0 < \mu \leqslant \mu_2$. The number of zeros of the equation (4.2) within the circle $|p - \rho^*(0)| \leqslant \varepsilon$ will be equal to the multiplicity of the root $\rho^*(0)$. These roots will tend to $\rho^*(0)$ as $\mu \to 0$.

If there exist non-commensurate numbers among the numbers $i\alpha_1$, $i\alpha_2$, ..., $i\alpha_l$, then $\lim |\beta_r| = 0$ when $r \to +\infty$. In the series $L_0(p)S_0^*(p)$ the terms will have poles arbitrarily near to $\rho^*(0)$ for every $\mu \neq 0$. Therefore, the series (3.8) for $L_0(p)S_0^*(p)$ will diverge in the circle $|p - \rho^*(0)| \leqslant \varepsilon$ for every $\mu > 0$, $\varepsilon > 0$.

Nevertheless, the partial sum (3.8) for the series $L_0(p)S_0^*(p)$ from

$\sigma = 2$, to $\sigma = \sigma_0 < \infty$ is holomorphic in the circle $|p - \rho^*(0)| \leqslant \varepsilon$ for sufficiently small $\mu > 0$, $\varepsilon > 0$, and for any $\sigma_0 \geqslant 2$. This makes it possible to expand the roots of the equation (4.2) in formal series of increasing powers of μ (if the coefficients of the system (4.1) are analytic functions of μ when $|\mu| < \mu_1$).

For the system of equations (6.1) without lag in the argument, one can deduce the asymptotic nature of these expansions when $\mu \to 0$ on the basis of the work [7].

In the more general case of the system (4.1) with stationary lag in the argument, it is still possible to expand the roots of the equation (4.2) in formal power series of μ (these series usually diverge when $\mu \neq 0$). The asymptotic behavior of these expansions, when $\mu \to 0$, relative to the stability of the solutions has not yet been established.

Example 4.1. We shall find the characteristic exponent of the solution of the differential equation

$$\frac{d^2y(t)}{dt^2} + \mu^2 c\,\frac{dy(t-\tau)}{dt} + (\omega^2 + 2\mu\cos 2t)\,y(t) = 0 \tag{4.5}$$

where $\omega \neq k$ ($k = 0, 1, 2, \ldots$), $c > 0$, $\tau > 0$. From (1.8) we have

$$L_0(p) = p^2 + \mu^2 c p e^{-p\tau} + \omega^2, \quad L_1(p) = L_2(p) = \mu, \; l = 2, \; \alpha_1 = -\alpha_2 = 2i \tag{4.6}$$

Equation (4.2), (3.8) takes on the form

$$p^2 + \omega^2 + \mu^2 c p e^{-p\tau} - \frac{\mu^2}{(p+2i)^2+\omega^2} - \frac{\mu^2}{(p-2i)^2+\omega^2} + O(\mu^4) = 0 \tag{4.7}$$

From equation (4.7) we find the approximate value p

$$p = i\omega + \frac{i\mu^2}{4\omega(1-\omega^2)} + \frac{i\mu^2 c\sin\tau\omega}{2} - \frac{\mu^2 c\cos\tau\omega}{2} + O(\mu^4) \tag{4.8}$$

For sufficiently small values of $|\mu|$ the solutions of (4.5) will be asymptotically stable when

$$c\cos\tau\omega > 0 \tag{4.9}$$

and unstable when $c\cos\tau\omega < 0$.

Example 4.2. Let us find the approximate equation of the boundary of the region of instability of the solutions of the differential equation with almost periodic coefficients with a lag in the argument

$$\frac{d^2y(t)}{dt^2} + \mu c\,\frac{dy(t)}{dt} + \lambda y(t) + 2\mu\sum_{q=1}^{l} b_q\cos\omega_q t\,y(t-\tau_q) = 0 \tag{4.10}$$

Here, $c > 0$, $\mu > 0$, $\lambda \approx 0$, $\mu \approx 0$; c, λ, $\omega_q > 0$, and $\tau_q \geqslant 0$ are real numbers. From (1.8) we obtain

$$L_0(p) = p^2 + \mu c p + \lambda, \qquad L_{2q-1}(p) = L_{2q}(p) = \mu b_q \exp(-\tau_q p) \tag{4.11}$$

Equation (4.2) takes on the form

$$p^2 + \mu c p + \lambda - \mu^2 \sum_{q=1}^{l} \left(\frac{b_q{}^2 \exp[-\tau_q(2p + i\omega_q)]}{(p + i\omega)^2} + \frac{b_q{}^2 \exp[-\tau_q(2p - i\omega_q)]}{(p - i\omega)^2} \right) + O(\mu^3) = 0 \tag{4.12}$$

Equation (4.12) has two roots p_1 and p_2 which become zero when $\mu \to 0$, $\lambda \to 0$. The left side of equation (4.12) is real when p is real and $\operatorname{Re} p_1 \neq \operatorname{Re} p_2$. Hence on the boundary of the region of instability $p = 0$.

We thus obtain the following equation [1] for the boundary of the region of instability:

$$\lambda = -2\mu^2 \sum_{q=1}^{l} b_q{}^2 \omega_q{}^{-2} \cos(\omega_q \tau_q) + O(\mu^3) \tag{4.13}$$

Example 4.3. Let us find the approximate expression for the characteristic exponent of the system of equations

$$\frac{dY(t)}{dt} = A + 2\mu \sum_{q=1}^{l} B_q \cos \omega_q t Y(t - \tau_q) = 0 \tag{4.14}$$

Here, $A = (a_1, a_2, \ldots, a_m)$ is a diagonal matrix. $\operatorname{Re} a_q \neq \operatorname{Re} a_s (q \neq s)$, $0 < \omega_1 < \omega_2 < \ldots < \omega_l$ are real numbers. Let us assume that $\alpha_{2q-1} = -\alpha_{2q} = i\omega_b$ $(q = 1, \ldots, l)$ in (4.1). From (1.8), and (4.14) we obtain

$$L_0(p) = Ep - A, \qquad L_{2q-1}(p) = L_{2q}(p) = -\mu B_q e^{-p\tau_q} \tag{4.15}$$

Equation (4.2), (3.8) has the form

$$\operatorname{Det}\left(L_0(p) - L_0(p) \sum_{q=1}^{n} (K_{2q-2}(p) K_{2q}(p + \omega_q i) + K_{2q}(p) K_{2q-1}(p - \omega_q i)) \right) + O(\mu^4) = 0 \tag{4.16}$$

From (1.7), (4.15) and (4.16) we obtain approximate expressions for the roots of the equations (4.16) which lie near to the a_q, $\operatorname{Im} a_q = 0$

$$p = a_k + 2\mu^2 \sum_{s=1}^{m} \sum_{q=1}^{n} b_{ks}^{(q)} b_{sk}^{(q)} \frac{(a_k - a_s)\cos \tau_q \omega_q - \omega_q \sin \tau_q \omega_q}{(a_k - a_s)^2 + \omega_q{}^2} + O(\mu^3) \tag{4.17}$$

Here the $b_{ks}^{(q)}$ are the elements of the matrix B_q, $B_q = \| b_{ks}^{(q)} \|_1^m$.

5. Let us consider the case of (1.1) when $\alpha_0 \equiv 0$, $\alpha_q = n_q \theta i$ ($q = 1, \ldots, l$), where the n_q are integers, i.e. we shall consider the case of a linear differential equation with periodic coefficients and stationary lag in the argument. The method of [4] is applicable to these equations. We shall give the most important results which follow from [4].

Theorem 5.1. Let $\alpha_0 \equiv 0$, $\alpha_q = n_q \theta i$ ($q = 1, 2, \ldots, l$) where the n_q are integers. In this case the representation (1.5) of $F(p)$, the transform of the solution $Y(t)$ of the system (1.1), can be continued analytically over the entire complex plane p. The components of the vector $F(p)$ are meromorphic functions of p which are regular and bounded if the Re p is sufficiently large. The poles of $F(p)$ are at points p_{jk} of the form

$$p_{jk} = p_j + k\theta i \qquad (j = 1, 2, 3, \ldots, \quad k = 0, \pm 1, \pm 2, \ldots) \tag{5.1}$$
$$(\operatorname{Re} p_j \to -\infty \text{ when } j \to +\infty)$$

Theorem 5.2. The general solution of the homogeneous ($\Phi(t) \equiv 0$) system of equations (1.1) with periodic coefficients and with stationary lag in the argument ($\alpha_0 \equiv 0$, $\alpha_q = n_q \theta i$, n_q integers) can be represented under the condition (1.2), as the asymptotic series ($t \to +\infty$)

$$Y(t) = \sum_{j=1}^{\infty} e^{p_j t} [B_{j0}(t) + tB_{j1}(t) + \ldots + t^{\delta_j} B_{j\delta_j}(t)] \tag{5.2}$$
$$(\operatorname{Re} p_1 \geqslant \operatorname{Re} p_2 \geqslant \operatorname{Re} p_3 \geqslant \ldots, \quad \operatorname{Re} p \to -\infty \text{ when } j \to +\infty)$$

The vectors $B_{jk}(t)$ are regular in some strip along the real axis t, and are periodic of period $2\pi\theta^{-1}$. If Re $p^* >$ Re p_r, then

$$\left\{Y(t) - \sum_{j=1}^{r} e^{p_j t} [B_{j0}(t) + tB_{j1}(t) + \ldots + t^{\delta_j} B_{j\delta_j}(t)]\right\} e^{-p^* t} \underset{t\to\infty}{\to} 0 \tag{5.3}$$

A special case of Theorem 5.2 was proved in [6].

Note 5.1. The series (3.8) in (4.2) can be continued analytically over the entire complex plane p using [4, p.595, Lemma 7.1].

Note 5.2. If in the system (4.1) the coefficients are regular functions of μ when $|\mu| \leqslant \mu$, the roots of the equation (4.2) can be expanded in series of increasing powers (in general fractional) of μ. These series will converge when $0 < |\mu| \leqslant \varepsilon$, $\varepsilon > 0$, but they can contain negative powers of μ. Suppose that when $\mu = 0$ the system of differential equations (4.1) does not contain terms with a lag in the argument, and has the characteristic exponents p_j^0 ($j = 1, 2, \ldots, m \times n$). For sufficiently small values $|\mu| \leqslant \varepsilon$, $\varepsilon > 0$, the characteristic exponents $p_j(\mu)$ which have the largest real part are arbitrarily near to p_j^0. The functions $p_j(\mu)$ will be bounded when $|\mu| \leqslant \varepsilon$. If the p_j^0 satisfy the condition

$$p_j^0 - p_h^0 \neq k\theta \qquad (h = 1, \ldots, m \times n, h \neq j,\ k = 0, \pm 1, \pm 2, \ldots) \tag{5.4}$$

then $dp_j(\mu)/d\mu = 0$ when $\mu = 0$.

6. In this section we extend the method of [4] to equations with almost periodic coefficients and with a lag in the argument.

Let $r_0, r_1, \ldots, r_\eta, k_0, k_1, \ldots, k_\xi$ be non-negative integers. We introduce the matrix-functions $S(p)$ by means of a matrix series [4, p.590] of the form

$$S_{k_0, k_1, \ldots, k_\xi}^{r_0, r_1, \ldots, r_\eta}(p) = \sum_{\sigma=1}^{\infty} \sum_{\varkappa_\sigma} K_{q_1}(p + \beta_{k_0}) K_{q_2}(p + \beta_{k_0} + \alpha_{q_1}) \times$$
$$\times K_{q_3}(p + \beta_{k_0} + \alpha_{q_1} + \alpha_{q_2}) \ldots K_{q_\sigma}(p + \beta_{k_0} + \alpha_{q_1} + \ldots + \alpha_{q_{\sigma-1}}) \tag{6.1}$$

The letter $\varkappa_\sigma$ denotes various sets of indices $q_j = 1, \ldots, l$ $(j = 1, \ldots, \sigma)$ satisfying the auxiliary conditions

$$\text{(a)} \qquad \beta_{k_0} + \alpha_{q_1} + \alpha_{q_2} + \ldots + \alpha_{q_\sigma} = \beta_{r_0} \tag{6.2}$$

$$\text{(b)} \qquad \{\beta_{k_1}, \beta_{k_2}, \ldots, \beta_{k_\xi}\} \cap \{\beta_{k_0} + \alpha_{q_1}, \beta_{k_0} + \alpha_{q_1} + \alpha_{q_2}, \ldots, \beta_{k_0} + \alpha_{q_1} + \ldots + \alpha_{q_{\sigma-1}}\} = \Lambda$$

$$\text{(c)} \qquad \{\beta_{r_1}, \beta_{r_2}, \ldots, \beta_{r_\eta}\} \subset \{\beta_{k_0} + \alpha_{q_1}, \beta_{k_0} + \alpha_{q_1} + \alpha_{q_2}, \ldots, \beta_{k_0} + \alpha_{q_1} + \ldots + \alpha_{q_{\sigma-1}}\}$$

Here, $k_0, k_1, \ldots, k_\xi, r_0, r_1, \ldots, r_\eta$ denote the ordinals of the numbers β_r introduced in Section 2. The symbols $\{\ \}$ denote a set; $\cap$ is the symbol indicating the intersection of sets; $\subset$ is the inclusion sign for sets; Λ is the null set of indices; the $K_q(p)$ are the matrices from (1.7).

Making use of the notation (6.1), the series (1.11) can be written in the form

$$F(p) = \Omega(p) + \sum_{r=0}^{\infty} S_0^r(p)\, \Omega(p + \beta_r) \tag{6.3}$$

From (6.2) it follows that the larger the lower indices, the "better" will be the convergence of the series (6.1), in the sense that terms that do not satisfy the auxiliary condition (b) of (6.2) will drop out of the series (6.1).

As was done in [4], one can prove relations which generalize the Lemma (7.1 of [4, p. 595] for functions $S(p)$ of the definition (6.1). These relations make it possible to express the matrix-function $S(p)$ by means of a matrix-function with an additional lower index γ.

We now give the final formulas for the three possible cases·

(A) Suppose $\gamma = k_0, \neq k_1, k_2, \dots, k_\alpha$; then

$$S^r_{k_0, k_1, \dots, k_\alpha}(p) = (E - S^{k_0}_{k_0, k_1, \dots, k_\alpha, \gamma}(p))^{-1} S^r_{k_0, k_1, \dots, k_\alpha, \gamma}(p) \tag{6.4}$$

(B) Suppose $\gamma = r \neq k_0, k_1, \dots, k_\alpha$, then

$$S^r_{k_0, k_1, \dots, k_\alpha}(p) = S^r_{k_0, k_1, \dots, k_\alpha, \gamma}(p)(E - S^r_{r, k_1, \dots, k_\alpha, \gamma}(p))^{-1} \tag{6.5}$$

(C) Suppose $\gamma \neq r, k_0, k_1, \dots, k_\alpha$, then

$$\begin{aligned} S^r_{k_0, k_1, \dots, k_\alpha}(p) &= S^r_{k_1, k_1, \dots, k_\alpha, \gamma}(p) + \\ + S^\gamma_{k_0, k_1, \dots, k_\alpha, \gamma}(p)&(E - S^\gamma_{\gamma, k_1, \dots, k_\alpha, \gamma}(p))^{-1} S^r_{\gamma, k_1, \dots, k_\alpha, \gamma}(p) \end{aligned} \tag{6.6}$$

If the numbers α_q are commensurate (Section 5), then these relations make it possible to analytically continue the series (1.11), (6.3) over the entire complex plane p. This leads to Theorem 5.1.

The equation (4.2) for the determination of the singularities of the representation $F(p)$ takes on the form

$$\operatorname{Det} D(p) \equiv \operatorname{Det}(L_0(p) - L_0(p) S^0_{0,0}(p)) = 0 \tag{6.7}$$

in the notation of (6.1).

Let us consider the problem on the stability of the solutions of the system (4.1) in the case when the condition (4.3) is not satisfied. That is, let, e.g.

$$\rho_1(0) - \rho_0(0) = \beta_\gamma \, (\gamma \neq 0), \ \rho_k(0) - \rho_0(0) \neq \beta_r \, (k = 2, 3, \dots, r = 1, 2, \dots) \tag{6.8}$$

In this case it is convenient to use, in the solution of (6.7), a formula which follows from (6.6)

$$L_0(p) S^0_{0,0}(p) = L_0(p) S^0_{0,0,\gamma}(p) + L_0(p) S^\gamma_{0,0,\gamma}(p) \tag{6.9}$$

$$[L_0(p + \beta_\gamma) - L_0(p + \beta_\gamma) S^\gamma_{\gamma,0,\gamma}(p)]^{-1} L_0(p + \beta_\gamma) S^0_{\gamma,0,\gamma}(p)$$

Any finite partial sums of the series on the right-hand side of (6.9) are regular in p and μ within the region $|p - \rho_0(0)| \leqslant \varepsilon$, $|\mu| < \varepsilon$, when $\varepsilon > 0$ is small enough. The matrices $L_0^{-1}(p + \beta_r)(\beta_0 \equiv 0)$, and $L_0^{-1}(p+\beta_\gamma)$, i.e. those which have a singularity at the point $p = \rho_0(0)$, do not occur among the matrices of the form $L_0^{-1}(p + \beta_r)$ which enter into the series of the right-hand side of (6.9).

Example 6.1. Let us find the boundary of the region of instability of the solutions of the differential equation

$$\frac{d^2y(t)}{dt^2} + \lambda y(t) + 2\mu a \cos 2ty(t - \tau_1) + 2\mu b \cos 4ty\ (t - \tau_2) = 0 \tag{6.10}$$

when $\lambda \approx 1$, $\mu \approx 0$; λ, $\mu > 0$, $\tau_1 > 0$, $\tau_2 > 0$ are real parameters.

From (1.1) we have $l = 4$, $\alpha_0 \equiv 0$, $\alpha_1 = -\alpha_2 = 2i$, $\alpha_3 = -\alpha_4 = 4i$. From (1.8) we obtain

$$L_0(p) = p^2 + \lambda, \qquad L_1(p) = L_2(p) = \mu a e^{-p\tau_1}, \qquad L_3(p) = L_4(p) = \mu b e^{-p\tau_2} \tag{6.11}$$

Let us compute the numbers β_r, $\beta_0 \equiv 0$

$$\alpha_1 = \beta_1 = 2i, \quad \alpha_2 = \beta_2 = -2i, \quad \alpha_3 = \beta_3 = 4i, \quad \alpha_4 = \beta_4 = -4i, \quad 2\alpha_1 = \beta_3$$
$$\alpha_1 + \alpha_2 = \beta_0, \quad \alpha_1 + \alpha_3 = \beta_5 = 6i, \quad \alpha_1 + \alpha_4 = \beta_2, \quad 2\alpha_2 = \beta_4, \ldots \text{ etc.}$$

From (6.8) we find $\rho_1(0) = \sqrt{(-\lambda)} \approx i$, $\rho_2(0) \approx -i$, $\rho_2 - \rho_1 = -2i = \beta_2$. We construct linear combinations of the form (2.11) which yield β_0 and β_2. We obtain

$$\alpha_1 + \alpha_2 = \alpha_2 + \alpha_1 = \alpha_3 + \alpha_4 = \alpha_4 + \alpha_3 = \ldots = \beta_0 \tag{6.12}$$
$$\alpha_2 = \alpha_1 + \alpha_4 = \alpha_4 + \alpha_1 = \ldots = \beta_2$$

The combination $\alpha_2 + \alpha_1$ does not satisfy the condition (b) of (6.2) for the function $S^0_{0,0,\gamma}(p)$.

The equation (6.7), in combination with (6.9), takes on the form

$$\begin{aligned}
&\{L_0(p) - L_0(p)[K_1(p)K_2(p+2i) + K_3(p)K_4(p+4i) + K_4(p)K_3(p-4i) + \ldots]\} \times \\
&\times \{L_0(p-2i) - L_0(p-2i)[K_2(p-2i)K_1(p-4i) + K_3(p-2i)K_4(p+2i) + \\
&\qquad + K_4(p-2i)K_3(p-6i) + \ldots]\} = \\
&= \{L_0(p)[K_2(p) + K_1(p)K_4(p+2i) + K_4(p)K_1(p-4i) + \ldots]\} \times \\
&\times \{L_0(p-2i)[K_1(p-2i) + K_3(p-2i) + K_2(p-2i) + K_2(p-2i)K_3(p-4i) + \ldots]\}
\end{aligned} \tag{6.13}$$

In (6.13) all series are written out with an accuracy within infinitesimals of the order $O(\mu^2)$, inclusive. Making use of the scalar nature of the functions $L_q(p)$ (6.11), we could eliminate the denominator in (6.7) and (6.9) and obtain the equation (6.13). After substitution of the expressions (1.7) (6.11) into (6.13), we obtain an explicit equation for the characteristic exponents p. From the condition of the negativeness of the real parts of the roots of the equation (6.13) near to i, we obtain

$$\left(\lambda-1+\frac{\mu^2a^2}{8}\cos 4\tau_1+\frac{\mu^2b_1^2}{6}\cos^2 2\tau_2\right)^2>\left(\mu a+\frac{\mu^2ab}{4}\cos 2(\tau_2-\tau_1)\cos(\tau_2+2\tau_1)\right)^2+O(\mu^4) \tag{6.14}$$

The condition (6.14) is only necessary, but it is not sufficient for the stability of the solutions of (6.10).

7. We shall try to find an asymptotic (with $\mu\to 0$) criterion of stability of the solutions of a linear differential equation with almost periodic coefficients [7]

$$(1+\mu f_2(t))\frac{d^2y}{dt^2}+\mu f_1(t),\quad \frac{dy}{dt}+(\lambda+\mu f_0(t))\,y=0 \tag{7.1}$$

Here, λ, $\mu \geqslant 0$ are real parameters; the $f_k(t)$ are real functions

$$f_k(t)=\sum_{q=0}^{l}a_{qk}(\mu)\,e^{-\alpha_q t}\quad (k=0,1,2),\qquad \alpha_0\equiv 0,\qquad \alpha_q=\theta_q i\quad (\theta_q\neq\theta_h(q\neq h)) \tag{7.2}$$

where $a_{qk}(u)$ are sufficiently often differentiable functions of μ, and the θ_q ($q=1,\ldots,l$) are arbitrary real numbers. From (1.8) we obtain

$$L_0(p)=(1+\mu a_{02})\,p^2+\mu a_{01}p+\lambda+\mu a_{00},\quad L_q(p)=\mu\,(a_{q2}p^2+a_{q1}p+a_{q0}) \tag{7.3}$$

Just as was done in [4, p.598], we form the equation (6.7) for the determination of the characteristic exponents. The conditions of the negativeness of the real parts, and the application of the results of [7] lead to the next theorem.

Theorem 7.1. Let the "resonance" case be given when $-2i\sqrt{\lambda}=\beta_\gamma\neq 0$. In order that the solutions of (7.1) be asymptotically stable for sufficiently small values of $\mu(0<\mu\leqslant\varepsilon_1)$, it is sufficient that the next two conditions be fulfilled when $0<\mu\leqslant\varepsilon_1$:

$$h(\mu)\equiv\lim_{\tau\to\infty}\mu\tau^{-1}\int_0^\tau\frac{f_1(t)}{1+\mu f_2(t)}\,dt>0 \tag{7.4}$$

$$g_\gamma(\mu,\lambda)\equiv|c_\gamma(i\sqrt{\lambda})|^2-|d_\gamma(i\sqrt{\lambda})|^2>0 \tag{7.5}$$

The solutions of (7.1) will be unstable if $h(\mu) < 0$ or $g_\gamma(\mu, \lambda) < 0$. If $h(\mu) = 0$, or $g(\mu, \lambda) = 0$, we have the doubtful case.

Here we have used the notations of (6.1)

$$c_\gamma(p) = L_0(p) - L_0(p) S^0_{0,0,\gamma}(p), \qquad d_\gamma(p) = L_0(p) S^\gamma_{0,0,\gamma}(p) \quad (7.6)$$

Note 7.1. If $-2i\sqrt{\lambda} \neq \beta_\gamma$ $(\gamma = 1, 2, \ldots)$ in (7.1), then the solutions of (7.1) are stable when $h(\mu) > 0$, and they are unstable when $h(\mu) < 0$. The stability is considered in the asymptotic sense when $\mu \to 0$.

Note 7.2. The series in (7.5) diverge when $\mu > 0$. The inequality (7.5) is taken in the asymptotic sense when $\mu \to 0$, $\mu > 0$. It is considered to be satisfied if the first non-zero coefficient of the expansion $g_\gamma(\mu, \lambda)$ in powers of μ is positive.

Note 7.3. The condition (7.5) can be found from the condition of the existence of an almost periodic solution of the equation (7.1).

Example 7.1. Let us consider the stability of the solutions of the equation

$$\frac{d^2y}{dt^2} + \mu^n c \frac{dy}{dt} + (\lambda + 2\mu\cos\omega_1 t + 2\mu\cos\omega_2 t)\, y = 0 \qquad \begin{pmatrix} c > 0 \\ \mu > 0 \end{pmatrix} \quad (7.7)$$

where ω_1, ω_2 are rationally non-commensurate real numbers. From (1.1) we have the case when $\alpha_1 = -\alpha_2 = i\omega_1$, $\alpha_3 = -\alpha_4 = i\omega_2$. The resonance values $\lambda_\gamma = -0.25\,\beta_\gamma^2$ form by (2.11) a denumerable and everywhere dense (when $\lambda > 0$) set of numbers of the form

$$\lambda_\gamma = -0.25\beta_\gamma^2 = 0.25\,(k_1\omega_1 + k_2\omega_2)^2 \qquad (k_1, k_2 = 0, \pm 1, \pm 2, \ldots) \quad (7.8)$$

When $c > 0$, one can attach to the axis $\mu = 0$ only a finite number of regions of instability. The order of the width of the regions of instability (7.7) which touch λ_γ, is equal to $O(\mu^{|k_1| + |k_2|})$.

From (2.11) we obtain

$$\beta_0 \equiv 0, \quad \beta_1 = i\omega_1, \quad \beta_2 = -i\omega_1, \quad \beta_3 = i\omega_2, \quad \beta_4 = -i\omega_2, \quad \beta_5 = 2i\omega_1, \quad \beta_6 = i(\omega_1 + \omega_2), \ldots$$

Let

$$\lambda \approx \lambda_1 = 0.25\,\omega_1^2, \quad \lambda \approx \lambda_5 = \omega_1^2, \quad \lambda \approx \lambda_6 = 0.25\,(\omega_1 + \omega_2)^2$$

The nontrivial condition of stability (7.5) takes on the form

$$\left(\lambda - 0.25\omega_1^2 + \frac{\mu^2}{2\omega_1^2} + \frac{2\mu^2}{\omega_2^2 - \omega_1^2}\right)^2 - \mu^2 + O(\mu^4) > 0 \tag{7.9}$$

$$\left(\lambda - \omega_1^2 - \frac{2\mu^2}{3\omega_1^2} + \frac{2\mu^2}{\omega_2^2 - 4\omega_2^2}\right)^2 - \frac{\mu^4}{\omega_1^4} + O(\mu^6) > 0 \tag{7.10}$$

$$\left(\lambda - 0.25(\omega_1 + \omega_2)^2 - \frac{2\mu^2}{\omega_1(\omega_1 + 2\omega_2)} - \frac{2\mu^2}{\omega_2(\omega_2 + 2\omega_1)}\right)^2 - \frac{4\mu^4}{\omega_1^2\omega_1^2} + O(\mu^6) > 0 \tag{7.11}$$

If one considers the equation

$$\frac{d^2y}{dt^2} + (\lambda + \mu f(t))\, y = 0 \tag{7.12}$$

where $f(t)$ is a real function of the form

$$f(t) = \sum_{q=1}^{l} (a_q \cos\omega_q t + b_q \sin\omega_q t) \tag{7.13}$$

then, on the basis of Theorem 7.1, one can prove that to every resonance value λ_γ, $\mu = 0$

$$\lambda_\gamma = -0.25\,\beta_\gamma^2 = 0.25\,(\omega_1 k_1 + \omega_2 k_2 + \ldots + \omega_l k_l)^2 \tag{7.14}$$
$$(k_q = 0, \pm 1, \pm 2, \ldots)$$

one can attach (for $\mu > 0$) a region of asymptotic instability of the solutions. The width of the region of instability will hereby be of the order $O(\mu^{\mathrm{Rg}\beta_\gamma})$.

The symbol $\mathrm{Rg}\,\beta_\gamma$ here is given by

$$\mathrm{Rg}\,\beta_\gamma = \min(|k_1| + |k_2| + \ldots + |k_l|) \tag{7.15}$$

under the condition that

$$i(\omega_1 k_1 + \omega_2 k_2 + \ldots + \omega_l k_l) = \beta_\gamma$$

The problem of sufficient conditions for the stability of the solutions of the equation (7.12) has not been solved as yet for the general case, to the knowledge of the author. The method proposed in this section for the investigation of the stability of the solutions of (7.1) does not contain any new principles if it is compared to [7], but it is more convenient to use in concrete computations.

Example 7.2. The solutions of the differential equation

$$\frac{d^2y}{dt^2} + 2\mu\cos\omega_1 t\,\frac{dy}{dt} + (\lambda + 2\mu\cos\omega_2 t)\, y = 0 \tag{7.16}$$

where $\mu > 0$, and ω_1, ω_2 are rationally non-commensurate and are not stable when $\lambda \approx 0.25\ \omega_1^2$ if

$$\left(\lambda - \frac{\omega_1^2}{4} - \frac{3}{8}\mu^2 + \frac{2\mu^2}{\omega_2^2 - \omega_1^2} + O(\mu^4)\right)^2 < \left(\frac{\mu\omega_1^2}{2} + O(\mu^3)\right)^2 \qquad (7.17)$$

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